F. The Fermi Temperature T_F sets Fermi Gas' Temperature Scale

- $T = 0K$ Physics $\Rightarrow$ fermions stack up in s.p. states $\Rightarrow$ filled up to $\varepsilon = E_F$

$$E_F = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3} \equiv kT_F \Rightarrow T_F = \frac{1}{k}\frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3} \sim \left(\frac{N}{V}\right)^{2/3}$$

[Metals : $E_F \sim$ few eV, $\underbrace{T_F \sim \text{few } 10^4 K}_{\text{so metal physics is low-temperature physics}}$]

Key Physics is that Fermions observe the $\underbrace{\text{Pauli Exclusion Rule}}_{\text{OR wavefunction requirement in QM}}$

- When would fermions don't need to care about their Quantum character?

We know that all gases (fermi/Bose gases) approaches classical ideal gas at high temperature and dilute (small $\frac{N}{V}$) limits

Recall: Classical Ideal Gas behavior requires (see Ch. IX)

$$\left(\frac{V}{N}\right)\frac{1}{\lambda_{th}^3(T)} >> 1 \quad \text{OR} \quad \underbrace{\left(\frac{V}{N}\right)^{1/3}}_{\text{spacing between particles in system}} >> \underbrace{\lambda_{th}(T)}_{\frac{h}{\sqrt{2\pi m k T}}} \qquad (20)$$

$\frac{h}{\sqrt{2\pi mkT}}$ thermal de Broglie wavelength

Particles NOT crowded

⇒ No need to worry about quantum nature

The criterion also sets a criterion on temperature

$$\frac{h}{\sqrt{2\pi m k T_0}} \approx \left(\frac{V}{N}\right)^{1/3}$$

$$h\left(\frac{N}{V}\right)^{1/3} \approx \sqrt{2\pi m k T_0}$$

$$\text{OR} \quad T_0 \sim \frac{h^2}{2\pi m}\frac{1}{k}\left(\frac{N}{V}\right)^{2/3} \qquad (21)$$

$\boxed{\text{we need to worry about quantum nature of particles when } T \lesssim T_0}$

For Fermions, $T_0 \approx T_F$

for electrons (m_e is tiny), $T_F \sim \frac{\hbar^2}{2m_e}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$ very high temperature
($\frac{N}{V}$ is high in metals)

(i) T_F is also the temperature below which quantum nature of fermions matters (this is why we need to use $f_{FD}(\varepsilon)$ to study Fermi Gas physics)

(ii) Only when $T > T_F$ (or $T \gg T_F$), behavior of Fermi Gas will approach+ that of classical gas

[note: metals would have melted at such temperatures]

+ See next section for behavior of Fermi Gas for $T \gg T_F$

For Bosons

- Argument in getting $T_0 \approx \frac{h^2}{2m\pi k}\left(\frac{N}{V}\right)^{2/3}$ does NOT rely on fermions or bosons
- Some $\underbrace{\text{atoms}}_{\text{nucleus + electrons}}$ are bosons

$$\begin{cases} m \to m_{atom} \gg m_e \quad (10^3 \text{ times higher}) \\ \text{atom gases are dilute } \left(\frac{N}{V}\right) \ll n_{\text{electron in metals}} \end{cases}$$

$\therefore T_0^{(bosons)}$ is typically very low

$T < T_0^{(bosons)} \Rightarrow$ Bosonic nature comes in (Bose-Einstein Condensation!)

$\hookrightarrow$ hard to observe because $T_0^{(boson)}$ is low and most gases would have turned into liquids/solids

G. Ideal Fermi Gas ain't classical ideal gas even at high temperature

[This section is Optional]

Question: What is the behavior of Ideal Fermi Gas at high temperature?

What for? All gases approach classical ideal gas at high temperatures, but any signature of fermions in high-T behavior?

Classical ideal gas $pV = NkT$ or $\frac{p}{kT} = \frac{N}{V}$

Fermi Gas at high temperature $\frac{pV}{NkT} = 1 + (\text{something})$?

or $$\frac{p}{kT} = \frac{N}{V} + \underbrace{(\text{some factor})}_{\text{What is this?}}\left(\frac{N}{V}\right)^2$$

Approach

- We have Eqs. (1), (2), (3), (4) [Sec. A] $g_s = 2$ used
- Eq. (4) says $$pV = \frac{2}{3}E = \frac{2}{3}\underbrace{\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}}_{\mathcal{A}}\int_0^\infty \frac{\varepsilon^{3/2}}{e^{(\varepsilon-\mu)/kT}+1}\,d\varepsilon$$

[Call $x = \varepsilon/kT$, $\varepsilon^{3/2} = (kT)^{3/2}x^{3/2}$, $d\varepsilon = (kT)\,dx$, $\varepsilon^{1/2} = (kT)^{1/2}x^{1/2}$]

$$pV = \frac{2}{3}\mathcal{A}(kT)^{5/2}\underbrace{\int_0^\infty \frac{x^{3/2}}{e^{-\mu/kT}e^{x}+1}\,dx}_{\text{after integrating over } x,\text{ it is a function of } e^{-\mu/kT}} \qquad (22) \qquad \text{(Exact)}$$

Similarly, Eq. (1) says

$$N = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}}{e^{(\varepsilon-\mu)/kT}+1}\,d\varepsilon = \mathcal{A}(kT)^{3/2}\underbrace{\int_0^\infty \frac{x^{1/2}}{e^{-\mu/kT}e^{x}+1}\,dx}_{\text{after integrating over } x,\text{ it is a function of } e^{-\mu/kT}} \qquad (23) \qquad \text{(Exact)}$$

Let's call $e^{+\mu/kT} \equiv \xi$, so $e^{-\mu/kT} = \xi^{-1}$ (24)

Define
$$\boxed{\int_0^\infty \frac{x^{n-1}}{\xi^{-1}e^x+1}\,dx = \underbrace{\Gamma(n)}_{\text{Gamma Function}^+}\cdot f_n(\xi)} \tag{25}$$

Note:
$\Gamma(1/2) = \sqrt{\pi}$
as $\Gamma(n) = \int_0^\infty z^{n-1}e^{-z}\,dz$
$\Gamma(3/2) = 1/2\,\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2}$
$\Gamma(5/2) = 3/2\,\Gamma(3/2) = \frac{3}{4}\sqrt{\pi}$

Using Eqs. (22), (23),

$$\frac{pV}{N} = \frac{2}{3}kT\,\frac{\int_0^\infty \frac{x^{3/2}}{\xi^{-1}e^x+1}\,dx}{\int_0^\infty \frac{x^{1/2}}{\xi^{-1}e^x+1}\,dx}$$

$$\frac{pV}{NkT} = \frac{2}{3}\,\frac{\Gamma(5/2)\,f_{5/2}(\xi)}{\Gamma(3/2)\,f_{3/2}(\xi)} = \cancel{\frac{2}{3}\,\frac{\frac{3}{4}\sqrt{\pi}}{\frac{1}{2}\sqrt{\pi}}}\,\frac{f_{5/2}(\xi)}{f_{3/2}(\xi)} = \frac{f_{5/2}(\xi)}{f_{3/2}(\xi)}$$

this is "1"

$$\Rightarrow \boxed{\frac{pV}{NkT} = \frac{f_{5/2}(\xi)}{f_{3/2}(\xi)}} \tag{26}$$

(Exact) (True for all temperatures)

+ See Notes on the Gamma Functions under "Essential Math Skills"

So far, all equations are exact, just rewriting Eqs. (1) - (4).

If we know ξ ($\xi = e^{+\mu/kT}$ (but μ itself can be negative as in classical gas)),

then Eq. (26) gives $\frac{pV}{NkT}$ (more than just "1").

Recall $\mu(T)$ (thus $\xi = e^{\mu/kT}$) is fixed by the "N-equation" (Eq. (1) & Eq. (23))

i.e.

$$N = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(kT)^{3/2}\ \underbrace{\Gamma(3/2)}_{\frac{\sqrt{\pi}}{2}}\ f_{3/2}(\xi)$$

g_s (spin degeneracy)

$$= 2\,\frac{V}{4\pi^2}\left(\frac{\sqrt{2mkT}}{\hbar}\right)^3\cdot\frac{\sqrt{\pi}}{2}\ f_{3/2}(\xi)$$

$$= 2\,V\left(\frac{\sqrt{2\pi mkT}}{h}\right)^3 f_{3/2}(\xi) = 2\,\frac{V}{\lambda_{th}(T)^3}\,f_{3/2}(\xi)$$

$$\boxed{N = 2\,\frac{V}{\lambda_{th}^3}\,f_{3/2}(\xi)} \quad (27)$$

(Exact) can be used to fix ξ

Summary

$$\frac{pV}{NkT} = \frac{f_{5/2}(\xi)}{f_{3/2}(\xi)} \quad ; \quad N = 2\frac{V}{\lambda_{th}^3} f_{3/2}(\xi) \quad ; \quad \xi = e^{\mu/kT}$$

advanced-level way of writing the governing equations of 3D ideal Fermi gas

$$f_n(\xi) = \frac{1}{\Gamma(n)} \int_0^{\infty} \frac{x^{n-1}}{\xi^{-1} e^{x} - 1} dx$$

They are <u>exact</u>, and applicable to all temperature.

Back to our goal, what is behavior in approach the classical gas limit?

Recall: Obtained for classical ideal gas [See Application CE-C7]

$$\mu = -kT \ln\left[\frac{V}{N}\left(\frac{\sqrt{2\pi mkT}}{h}\right)^3\right]$$

$$= \underset{\text{negative}}{-}kT \ln\left[\underbrace{\left(\frac{V}{N}\right)\frac{1}{\lambda_{th}^3}}_{\text{large}}\right]$$

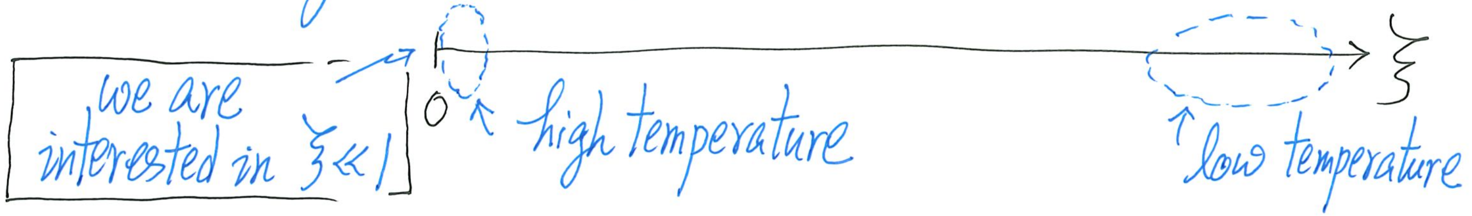
large $\left(\frac{V}{N}\right)^{1/3} >> \lambda_{th}$ (ideal gas)

$$\therefore \xi_{classical} = e^{\mu/kT} = e^{-\ln\left[\frac{V}{N}\frac{1}{\lambda_{th}^3}\right]} = \frac{\lambda_{th}^3}{\left(\frac{V}{N}\right)} \ll 1$$

As we want to study ideal Fermi Gas as it approaches the classical gas limit, we expect $\xi \approx \xi_{classical} \ll 1$ (28) ($T >> T_F$ situation)

Formally, $\mu(T=0) = E_F$, $\xi_{(T=0)} = e^{E_F/kT} \to \infty$; low-$T$, $\mu(T)$ shifts down a bit, ξ is large.

i.e. for ideal Fermi Gas $0 < \xi < \infty$



Strategy Update: Study $f_{5/2}(\xi)$ and $f_{3/2}(\xi)$ for $\underbrace{\xi \ll 1}$ ← small parameter for the regime we are interested in

$$\int_0^\infty \frac{x^{n-1}}{\xi^{-1}e^x+1}\,dx = \int_0^\infty \frac{\xi x^{n-1}e^{-x}}{1+\xi e^{-x}}\,dx \quad \text{(Exact)}$$

$$\approx \int_0^\infty \xi x^{n-1}e^{-x}\left[1-\xi e^{-x} + \overbrace{(\xi^2 \text{ and higher terms})}^{\text{ignored } (\xi \ll 1)}\right]dx$$

$$= \xi\int_0^\infty x^{n-1}e^{-x}\,dx - \xi^2\int_0^\infty x^{n-1}e^{-2x}\,dx$$

$$= \xi\,\Gamma(n) - \xi^2\int_0^\infty \frac{y^{n-1}}{2^{n-1}}e^{-y}\,\frac{dy}{2} \qquad \left(y=2x \text{ in 2nd term}\right)$$

$$= \xi\,\Gamma(n) - \frac{\xi^2}{2^n}\Gamma(n) = \Gamma(n)\underbrace{\left[\xi - \frac{\xi^2}{2^n}\right]}_{\text{this is } f_n(\xi) \text{ for } \xi \ll 1} = \Gamma(n)\overline{f_n(\xi)}^{\text{for } \xi \ll 1}$$

$$\therefore \boxed{f_n(\xi) = \xi - \frac{\xi^2}{2^n}} \quad (29) \text{ for } \xi \ll 1$$

Recall: μ (or ξ now) is fixed by Eq. (1).

$$N = 2\,\frac{V}{\lambda_{th}^3(T)}\, f_{3/2}(\xi) \approx 2\,\frac{V}{\lambda_{th}^3}\left(\xi - \frac{\xi^2}{2^{3/2}}\right)$$

← spin degeneracy (usually not included in classical ideal gas calculations)

$$\Rightarrow \xi \approx \underbrace{\left(\frac{N}{V}\frac{\lambda_{th}^3}{2}\right)}_{\text{this is } \left(\frac{1}{2}\xi_{\text{classical}}\right) \ll 1 \text{, this is the leading term}} + \frac{\xi^2}{2^{3/2}}$$

$$\approx \left(\frac{N}{V}\frac{\lambda_{th}^3}{2}\right) + \frac{1}{2^{3/2}}\underbrace{\left(\frac{N}{V}\frac{\lambda_{th}^3}{2}\right)^2}_{\text{Even smaller!}}$$

$$\approx \underbrace{\left(\frac{N}{V}\frac{\lambda_{th}^3}{2}\right)}_{\text{just the classical ideal gas value}} \quad (30)$$

sufficient for our purpose

(temperature T inside $\lambda_{th} = \dfrac{h}{\sqrt{2\pi m k T}}$)

Eq. (26): $$\frac{pV}{NkT} = \frac{f_{5/2}(\xi)}{f_{3/2}(\xi)} \quad \text{(exact)} \quad \approx \frac{\xi - \frac{\xi^2}{2^{5/2}}}{\xi - \frac{\xi^2}{2^{3/2}}}$$ ($\xi \ll 1$, approaching classical gas limit)

$$\approx \left(1 - \frac{\xi}{2^{5/2}}\right)\left(1 + \frac{\xi}{2^{3/2}}\right) \approx 1 + \xi\left(\frac{1}{2^{3/2}} - \frac{1}{2^{5/2}}\right) + \text{ignored } (\xi^2, \xi^3 \ldots)$$

$$= 1 + \frac{1}{4\sqrt{2}}\xi$$

using ξ in Eq. (30)

$$= 1 + \underbrace{\frac{1}{4\sqrt{2}} \frac{\lambda_{th}^3}{2}}_{\text{something!}}\left(\frac{N}{V}\right) \quad (31)$$

$$\therefore \boxed{\frac{p}{kT} = \frac{N}{V} + \underbrace{\frac{1}{4\sqrt{2}} \frac{\lambda_{th}^3}{2}}\left(\frac{N}{V}\right)^2 = \frac{N}{V} + B_2(T)\left(\frac{N}{V}\right)^2} \quad (32)$$

$\therefore$ Ideal (Non-interacting) Fermi Gas behaves as $\frac{p}{kT} = n + B_2(T)n^2$ with a Positive Second Virial Coefficient $B_2(T) = \frac{1}{4\sqrt{2}} \frac{\lambda_{th}^3}{2}$

Positive $B_2(T)$ implies some effective repulsion between fermions, even though they don't interact!

Aside: We saw in Van der Waals Gas $\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$

attractive; repulsive; # moles

that

$$\frac{p}{kT} = n + \underbrace{B_2(T)}_{\left[+\frac{b}{N_A} - \frac{a}{N_A^2 kT}\right]} n^2$$

attractive part contributes negative term to B_2

repulsive part of particle-particle interaction contributes positive term to B_2

Summary

- Even when ideal Fermi Gas approaches the classical ideal gas limit, the first correction term indicates the quantum nature of the fermions with a positive B_2, as if they have an effective repulsive interaction.

Cross Reference: There is an analogous treatment for Ideal Bose gas with $B_2(T) = -\frac{1}{4\sqrt{2}}\frac{\lambda_{th}^3}{g_s}$, i.e. negative $B_2(T)$.

References

- Fermi Gas is a fascinating subject covered in most statistical mechanics books
- See Pathria, "Statistical Mechanics", Ch. 8

  Greiner, Neise, Stöcker, "Thermodynamics and Statistical Mechanics", Ch. 14

  [both provided a discussion on white dwarf stars]
- Ashcroft and Mermin, "Solid State Physics", Ch. 1, 2, 3

  [Sommerfeld model of metals and its failure]

Beyond Ideal Fermi Gas: Interacting Systems

Pathria (see above), Ch. 11

Diep, "Statistical Physics", Ch. 7 (Method of Second Quantization),
Ch. 10 (Systems of Interacting Electrons: Fermi Liquids)